JOURNAL OF MULTIVARIATE ANALYSIS 8, 262–267 (1978)

On Monotonicity of the Modified Likelihood Ratio Test for the Equality of Two Covariances*

M. S. SRIVASTAVA†

University of Toronto, Toronto, Canada

C. G. KHATRI

Gujarat University, India

AND

E. M. CARTER

University of Guelph

Communicated by P. R. Krishnaiah

For testing the hypothesis of equality of two covariances (Σ_1 and Σ_2) of two p-dimensional multivariate normal populations, it is shown that the power function of the modified likelihood ratio test increases as λ_1 increases from one and λ_r decreases from one where $\lambda_1 > \cdots > \lambda_r > 0$ are the distinct characteristic roots of $\Sigma_1\Sigma_2^{-1}$, $r \leqslant p$. As a by-product we get the unbiased result already established by Sugiura and Nagao (1968).

1. INTRODUCTION

Let $\mathbf{x}_{ij}$, $j = 1, 2,..., N_i$, $i = 1, 2$, be $(N_1 + N_2)$ independently distributed random vectors from $N_p(\boldsymbol{\mu}_i, \Sigma_i)$. Then the likelihood ratio test for the problem of testing the hypothesis H: $\Sigma_1 = \Sigma_2$ against the alternative A: $\Sigma_1 \neq \Sigma_2$ where $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ are unspecified has the acceptance region given by

$$\omega' = \{(V_1, V_2) : V_1 > 0, V_2 > 0 \text{ and } | V_1 |^{\frac{1}{2}N_1} | V_2 |^{\frac{1}{2}N_2} | V_1 + V_2 |^{-\frac{1}{2}(N_1+N_2)} \geqslant c_\alpha\}, \tag{1}$$

Received March 29, 1977; revised October 25, 1977.

AMS 1970 subject classification: 62H15.

Key words and phrases: Monotonicity, unbiasedness, modified likelihood ratio test, equality of covariances.

* Research supported by National Research Council of Canada.

† Currently visiting Math. Res. Center and Statistics Dept., University of Wisconsin, Madison, Wis.

0047-259X/78/0082-0262$02.00/0

where $V_i = \sum_{j=1}^{N_i} (\mathbf{x}_{ij} - \bar{\mathbf{x}}_i)(\mathbf{x}_{ij} - \bar{\mathbf{x}}_i)'$, $\bar{\mathbf{x}}_i = N_i^{-1} \sum_{j=1}^{N_i} \mathbf{x}_{ij}$, and c_α is so chosen that the error of the first kind is at a specified level α. Note that V_1 and V_2 are independently distributed as $W_p(\Sigma_1, n_1)$ and $W_p(\Sigma_2, n_2)$ respectively, where

$$n_i = N_i - 1, \qquad i = 1, 2. \tag{2}$$

For $p = 1$, Brown (1939) has shown that this acceptance region gives an unbiased test if and only if $N_1 = N_2$. Also, it can be obtained from Lehmann (1959, p. 170) that the uniformly most powerful unbiased test has the acceptance region ω obtained from ω' by replacing N_i with n_i (the degrees of freedom associated with V_i), this is called the modified likelihood ratio test and was proposed by Bartlett (1937). For general p, Sugiura and Nagao (1968) showed that the modified likelihood ratio test is unbiased. While the general monotonicity result is not yet available, we show in this note that the power of the modified likelihood ratio test increases if λ_1 increases from one and λ_r decreases from one, where $\lambda_1 > \lambda_2 > \cdots > \lambda_r > 0$ are the distinct roots of $\Sigma_1 \Sigma_2^{-1}$.

2. Preliminaries

To prove the monotonicity result we need the following two lemmas.

Lemma 1. *Let $D = \operatorname{diag}(I_{p_1}, 0)$ be a $p \times p$ diagonal matrix whose first p_1 diagonal elements are one and the rest are zeros. Then if $A = \operatorname{diag}(a_1 I_{p_1}, a_2 I_{p_2}, \ldots, a_r I_{p_r})$, where $a_1 > a_2 > \cdots > a_r > 0$ and $\sum_{i=1}^r p_i = p$,*

$$\begin{aligned} |I + AQ|^{-1} \frac{\partial}{\partial a_1} |I + AQ| &= \operatorname{tr} Q(I + AQ)^{-1} D \\ &= [p_1 - \operatorname{tr}(I + AQ)^{-1} D]/a_1 . \end{aligned}$$

The proof is straightforward.

Lemma 2. *Let the density function of a random p.d. matrix Q be*

$$h(Q \mid A) = \{B_p(f_1, f_2)\}^{-1} \mid A \mid^{f_1} \mid Q \mid^{f_1 - \frac{1}{2}(p+1)} \mid I + AQ \mid^{-f_1 - f_2}$$

for $Q > 0$, $f_1 > 0$ and $f_2 > 0$, where $A = \operatorname{diag}(a_1 I_{p_1}, a_2 I_{p_2}, \ldots, a_r I_{p_r})$, $a_1 > a_2 > \cdots > a_r > 0$, $\sum_{i=1}^r p_i = p$, $B_p(f_1, f_2) = \Gamma_p(f_1)\, \Gamma_p(f_2) / \Gamma_p(f_1 + f_2)$ and $\Gamma_p(n) = \pi^{\frac{1}{4}p(p-1)} \prod_{i=1}^p \Gamma(n - (i-1)/2)$. Let $g(a_1, a_2, \cdots, a_r) = P(|Q|^{f_1} \mid I + Q \mid^{-f_1 - f_2} \leqslant c)$ for any $0 < c < 1$. Then, for $a_1 \geqslant \tilde{a}_1 \geqslant \max(1, a_2)$,

$$g(a_1, a_2, \ldots, a_r) \geqslant g(\tilde{a}_1, a_2, \cdots, a_r)$$

while, for $a_r \leqslant \tilde{a}_r \leqslant \min(1, a_{r-1})$,

$$g(a_1, a_2, \cdots, a_r) \geqslant g(a_1, a_2, \cdots, a_{r-1}, \tilde{a}_r).$$

Proof. Let $g \equiv g(a_1, \cdots, a_r)$ and

$$\mathscr{D} = \{Q : |Q|^{f_1} |I + Q|^{-f_1-f_2} \leqslant c\}.$$

Then from Lemma 1, we get

$$\frac{\partial g}{\partial a_1} = (p_1 f_1/a_1)g - (f_1 + f_2)(p_1/a_1) \int_{\mathscr{D}} h(Q \mid A)[1 - p_1^{-1} \operatorname{tr}(I + AQ)^{-1} D]\, dQ. \tag{3}$$

Let

$$(I + AQ)^{-1} = (b_{ij}) \qquad \text{and} \qquad (I + AQ) = (c_{ij}).$$

Let us consider b_{11}, and b_{22}. By interchanging rows and columns of $I + AQ$, we can make c_{22} as the (1, 1)th element of $(I + AQ)$ without effecting the density and the region D. Hence b_{ii}'s $i = 1, 2, \ldots, p$, are identically distributed. Thus

$$\int_{\mathscr{D}} b_{ii} h(Q \mid A)\, dQ = \int_{\mathscr{D}} b_{11} h(Q \mid A)\, dQ, \qquad i = 1, 2, \ldots, p_1.$$

Hence (3) can be written as

$$\frac{\partial g}{\partial a_1} = (p_1 f_1/a_1)g - (p_1/a_1)(f_1 + f_2) \int_{\mathscr{D}} (1 - b_{11})\, h(Q \mid A)\, dQ. \tag{4}$$

Let

$$Q = \begin{pmatrix} q_{11} & \mathbf{q}' \\ \mathbf{q} & Q_{11} \end{pmatrix}, \qquad A = \begin{pmatrix} a_1 & \mathbf{0}' \\ \mathbf{0} & A_1 \end{pmatrix},$$

$$P_1 = Q_{11}^{-1} - (I + Q_{11})^{-1} = (Q_{11} + Q_{11}^2)^{-1},$$

and

$$P_2 = Q_{11}^{-1} - (A_1^{-1} + Q_{11})^{-1} = (Q_{11} + Q_{11} A_1 Q_{11})^{-1}.$$

The transformations

$$u = g_{11} - \mathbf{q}'(I + Q_{11})^{-1} \mathbf{q} \qquad \text{and} \qquad \mathbf{q} = u^{1/2}\mathbf{x}$$

give the following results

$$J \equiv J(q_{11}, \mathbf{q} \to u, \mathbf{x}) = J(q_{11} \to u)\; J(\mathbf{q} \to \mathbf{x}) = u^{(p-1)/2},$$

$$|Q| = |Q_{11}|\, (q_{11} - \mathbf{q}' Q_{11}^{-1} \mathbf{q}) = |Q_{11}|\, u(1 - \mathbf{x}' P_1 \mathbf{x}),$$

$$|I_p + AQ| = |I_{p-1} + A_1 Q_{11}|\, (1 + a_1\, \delta u),$$

$$(1 - b_{11}) = a_1\, \delta u/(1 + a_1\, \delta u),$$

and

$$\mathscr{D} = \mathscr{D}_1 \cap \mathscr{D}_2, \qquad \text{where} \quad \mathscr{D}_1 = \{Q_{11} > 0, \mathbf{x} : \mathbf{x}'P_1\mathbf{x} < 1\},$$
$$\mathscr{D}_2 = \{u : u^{f_1}(1+u)^{-f_1-f_2} \leqslant c_0\},$$

$$c_0 \equiv c_0(Q_{11}, \mathbf{x}) = c \mid I + Q_{11} \mid^{f_1+f_2} / \mid Q_{11} \mid^{f_1} (1 - \mathbf{x}'P_1\mathbf{x})^{f_1}, \tag{5}$$

and

$$\delta = 1 + \mathbf{x}'(P_2 - P_1)\,\mathbf{x}. \tag{6}$$

With the above notations, we have

$$a_1^{-1}(1 - b_{11})\, Jh(Q \mid A) = h_1(Q_{11}, \mathbf{x} \mid A_1)(a_1\delta)^{f_1}\, u^{f_1}/(1 + a_1\,\delta u)^{f_1+f_2+1},$$

$$h_1(Q_{11}, \mathbf{x} \mid A_1) = \frac{\mid A_1 \mid^{f_1} \mid Q_{11} \mid^{f_1 - \frac{1}{2}(p+1)} (1 - \mathbf{x}'P_1\mathbf{x})^{f_1-\frac{1}{2}(p+1)}}{B_p(f_1, f_2) \mid I_{p-1} + A_1 Q_{11} \mid^{f_1+f_2} \delta^{f_1-1}}$$

and hence (4) we can be written as

$$\frac{\partial g}{\partial a_1} = p_1 \int_{\mathscr{D}_1} h_1(Q_{11}, \mathbf{x} \mid A_1)\, dQ_{11}\, d\mathbf{x} \int_{\mathscr{D}_2} \left\{ \frac{\partial}{\partial u} \left[\frac{(a_1\delta)^{f_1-1}\, u^{f_1}}{(1 + a_1\,\delta u)^{f_1+f_2}} \right] \right\} du$$

$$= p_1 \int_{\mathscr{D}_1} h_1(Q_{11}, \mathbf{x} \mid A_1)(a_1\delta)^{f_1-1}\, [u^{f_1}/(1 + a_1\,\delta u)^{f_1+f_2}]_{\mathscr{D}_2}\, dQ_{11}\, d\mathbf{x}. \tag{7}$$

Now, there are exactly two solutions c_1 and c_2, $c_2 > c_1$, of the equation:

$$u^{f_1}(1 + u)^{-f_1-f_2} = c_0$$

and then,

$$\mathscr{D}_2 = \{0 \leqslant u \leqslant c_1\} \cup \{c_2 \leqslant u < \infty\}.$$

Therefore,

$$[u^{f_1}/(1 + a_1\,\delta u)^{f_1+f_2}]_{\mathscr{D}_2} = c_1^{f_1}(1 + a_1\,\delta c_1)^{-f_1-f_2} - c_2^{f_1}(1 + a_1\,\delta c_2)^{-f_1-f_2}$$

$$= c_2^{f_1}(1 + a_1\,\delta c_2)^{-f_1-f_2} \left[\left\{ \frac{(1 + c_1)(1 + a_1\,\delta c_2)}{(1 + c_2)(1 + a_1\,\delta c_1)} \right\}^{f_1+f_2} - 1 \right]$$

since $c_1^{f_1} = c_2^{f_2}[(1 + c_1)/(1 + c_2)]^{f_1+f_2}$. Now, since

$$a_1\delta = a_1 + a_1\mathbf{x}'[(Q_{11} + Q_{11}A_1Q_{11})^{-1} - (Q_{11} + Q_{11}^2)^{-1}]\mathbf{x}$$
$$= 1 + (a_1 - 1)(1 - \mathbf{x}'P_1\mathbf{x}) + \mathbf{x}'[a_1(Q_{11} + Q_{11}A_1Q_{11})^{-1}$$
$$- (Q_{11} + Q_{11}^2)^{-1}]\mathbf{x}$$

and since

$$\mathbf{x}'P_1\mathbf{x} \leqslant 1 \qquad \text{and} \qquad a_1(Q_{11} + Q_{11}A_1Q_{11})^{-1} - (Q_{11} + Q_{11}^2)^{-1} > 0,$$

we get

$$a_1\delta > 1 \qquad \text{if} \qquad a_1 \geqslant 1. \tag{9}$$

Using (9) and (8), we find that

$$[u^{f_1}/(1 + a_1\,\delta u)^{f_1+f_2}]_{\mathscr{D}_2} > 0 \qquad \text{if} \quad a_1 \geqslant 1,$$

and consequently (7) gives

$$\frac{\partial g}{\partial a_1} \geqslant 0 \qquad \text{if} \quad a_1 \geqslant 1 \qquad \text{and} \qquad a_1 > a_2 > \cdots > a_r\,. \tag{10}$$

Similarly, it can be established that

$$\frac{\partial g}{\partial a_r} \leqslant 0 \qquad \text{if} \quad a_r \leqslant 1 \qquad \text{and} \qquad a_1 > a_2 > \cdots > a_r\,. \tag{11}$$

These show that g increases if a_1 increases from one and a_r decreases from one.

3. Monotonicity

Theorem 1. *Let $\mathbf{y}_1 \sim N_p(\boldsymbol{\mu}_1, \Sigma_1)$, $V_1 \sim W_p(\Sigma_1, n_1)$, $\mathbf{y}_2 \sim N_p(\boldsymbol{\mu}_2, \Sigma_2)$ and $V_2 \sim W_p(\Sigma_2, n_2)$ be independently distributed, and let the distinct nonzero characteristic roots of $\Sigma_1\Sigma_2^{-1}$ be $\lambda_1 > \lambda_2 > \cdots > \lambda_r > 0$. Then, the power of the modified likelihood ratio test for testing $H\colon (\Sigma_1 = \Sigma_2)$ against $A\colon (\Sigma_1 \neq \Sigma_1)$ increases if λ_1 increases from one and λ_r decreases from one.*

Proof. The rejection region is given by

$$\omega = \{(V_1, V_2) : V_1 > 0, V_2 > 0, \mid V_1\mid^{\frac{1}{2}n_1} \mid V_2\mid^{\frac{1}{2}n_2} \mid V_1 + V_2\mid^{-\frac{1}{2}(n_1+n_2)} \leqslant c\}.$$

Since the problem and the rejection region are invariant under the group of nonsingular linear transformations $(V_1, V_2) \to (AV_1A', AV_2A')$, we assume without any loss of generality that $\Sigma_1 = I$ and $\Sigma_2 = A = \operatorname{diag}(\lambda_1 I_{p_1}, \ldots, \lambda_r I_{p_r})$, $p_1 + p_2 + \cdots + p_r = p$. Then, let $Q = V_2^{-1/2}V_1V_2^{-1/2}$. We now have the density of Q as $h(Q \mid A)$ and the power of the modified likelihood test as

$$g(\lambda_1, \ldots, \lambda_r) = \int_{\mathscr{D}} h(Q \mid A)\, dQ = P(Q \in \mathscr{D}),$$

where $h(Q \mid A)$ and $g(\lambda_1, \ldots, \lambda_r)$ are defined in Lemma 2. The result now follows from Lemma 2.

COROLLARY 1. *The modified likelihood ratio test in Theorem 1 is unbiased.*

Proof. Let the roots of $\Sigma_1\Sigma_2^{-1}$ be $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_p > 0$. Notice that here $a_1, a_2, ..., a_p$ may be distinct or equal. Let us suppose that the general alternative can be $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_i \geqslant 1 \geqslant a_{i+1} \geqslant \cdots \geqslant a_p > 0$ for $i = 0, 1, ..., p$. Let $g(a_1, a_2, ..., a_p) = P(Q \in \mathscr{D})$, Q being in the proof of Theorem 1. Then, decreasing from the maximum, a_1, to the next maximum, a_2, etc., we have

$$g(a_1, a_2, ..., a_p) \geqslant g(a_2, a_2, a_3, ..., a_p) \geqslant \cdots$$
$$\geqslant g(a_i, a_i, ..., a_{i+1}, ..., a_p) \geqslant g(1, 1, ..., 1, a_{r+1}, ..., a_p)$$

and now increasing from the minimum to the next minimum, etc., we have

$$g(1, ..., 1, a_{i+1}, ..., a_p) \geqslant g(1, ..., 1, a_{i+1}, ..., a_{p-2}, a_{p-1}, a_{p-1})$$
$$\geqslant \cdots \geqslant g(1, ..., 1, a_{i+1}, ..., a_{i+1}) \geqslant g(1, ..., 1)$$

and consequently

$$g(a_1, ..., a_p) \geqslant g(1, 1, ..., 1)$$

which proves the unbiasedness. Thus, Corollary 1 is established. Corollary 1 was established by Sugiura and Nagao (1968) by a different method.

REFERENCES

[1] BROWN, G. W. (1939). On the power of the L_1 test for the equality of several variances. *Ann. Math. Statist.* **10** 119–128.

[2] LEHMANN, E. L. (1959). *Testing Statistical Hypothesis.* Wiley, New York.

[3] SUGIURA, N., AND NAGAO, H. (1968). Unbiasedness of some test criteria for the equality of one or two covariance matrices. *Ann. Math. Statist.* **39** 1686–1692.